Play with These! Feb 12, 2011

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Here are several fun examples of open-ended investigations to try, as well as some games to investigate. We will discuss some of them today, but only scratch the surface. There are weeks and weeks of various inquiries below.

Remember that your inquiry is not complete until you understand *why*. Finding a pattern is the first step, but *proving* the pattern is essential.

Games

For all but #7, two players alternate turns. The winner is the last player who makes a legal move. See if you can find a winning strategy for one of the players. Try to prove that your strategy works. And, always, try to generalize!

1 *Takeaway*. A set of 16 pennies is placed on a table. Two players take turns removing pennies. At each turn, a player must remove between 1 and 4 pennies (inclusive).

2 *Variants/extensions of Takeaway*.

(a) Can you generalize the previous game to other values (besides 16 and 4)?

(b) What if a legal move was taking either 1 or 4 pennies (but no other values)?

(c) What if one of the pennies was glued to the table (and both players knew which penny it was)?

3 *Putdown*. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny that is already on the table. The table starts out completely bare.

4 *Puppies and Kittens*. We start with a pile of 7 kittens and 10 puppies. Two players take turns; a legal move is removing any number of puppies or any number of kittens or an equal number of both puppies and kittens.

5 *Color the Grids*. You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). A move is legal as long as no closed path has been created.

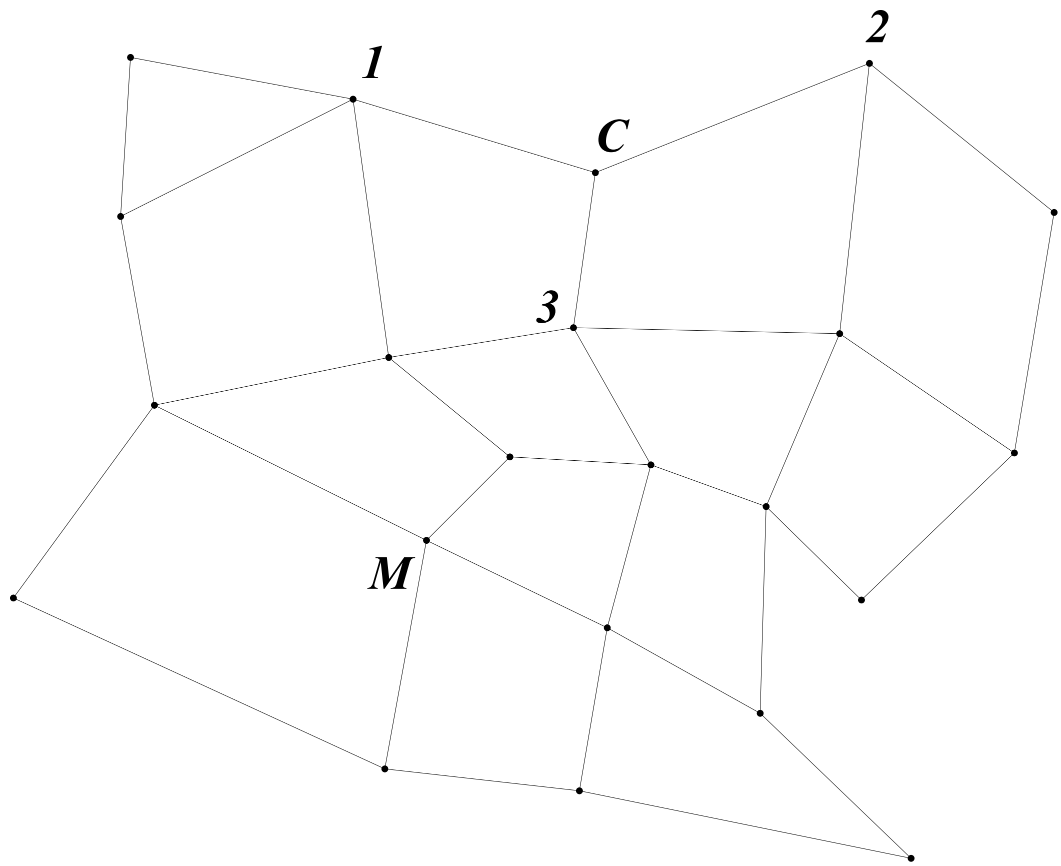
6 *Classic Nim*. Start with several piles of beans. A legal move consists of removing one or more beans from a pile.

(a) Verify that this game is *very* easy to play if you start with just one pile, for example, of 17 beans.

(b) Likewise, if the game starts with two piles, the game is quite easy to analyze. Do it!

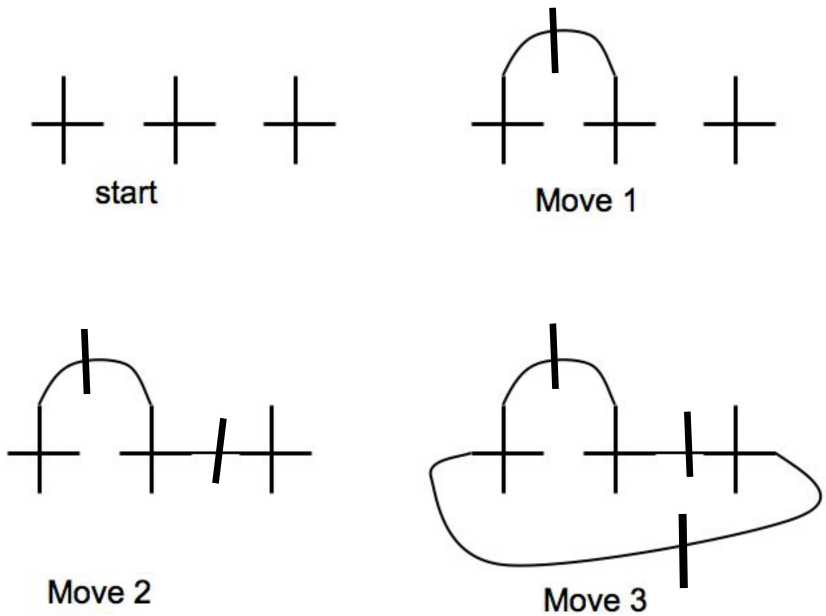
(c) But what if we start with three or more piles? For example, how do we play the game if it starts with three piles of 17, 11, and 8 beans, respectively? What about four piles? More?

7 *Cat and Mouse*. (Adapted from Ravi Vakil's book; see below.) A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win? Hint: expect the unexpected!

8 *Brussels Sprouts*. Start by putting a few crosses on a piece of paper. On each move, a player can connect the two endpoints of a cross together, with a single line (which can be curved). Then a new cross is drawn on this connection line. You cannot ever draw a line that intersects another already-drawn line. Here is an example of the first few moves of a game that starts with three crosses.



Other Investigations

1 The *triangular numbers* are the sums of consecutive integers, starting with 1. The first few triangular numbers are

$$1, \quad 1+2=3, \quad 1+2+3=6, \quad 1+2+3+4=10, \ldots.$$

Can a triangular number ever be a perfect square?

2 *Trapezoidal Numbers*. A larger class of numbers than triangular are the *trapezoidal* numbers, which are defined to be sums of two or more consecutive positive integers. Thus every triangular number (greater than 1) is automatically trapezoidal. But so to is $5 = 2+3$ and $303 = 100+101+102$. The question is, just which numbers are trapezoidal, and which numbers aren't trapezoidal?

3 *Jumping Frogs*. Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the "leapee" is at the midpoint of the line segment whose endpoints are the starting and ending position of the "leaper." Will a frog ever occupy the vertex of the square that was originally unoccupied?

4 Investigate the parity of the numbers in the Fibonacci sequence. Then move on to multiples of 3, 5, 7, etc.

5 Investigate the parity of the elements in Pascal's Triangle. What about divisibility by 3, 5, etc.? How many odd numbers are in row n? Do you see a pattern? Also, can you find Fibonacci numbers in Pascal's Triangle?

Resources for going further

General My all-time favorites are the works of Martin Gardner, most of which are compiled on just one CD. Gardner, who died recently at the age of 96, was the editor of the "Mathematical Games" column for *Scientific American* for many years.

There is a giant online community, with "AoPS" (artofproblemsolving.com) the undisputed leader.

Problem-Solving My book on problem solving, *The Art and Craft of Problem Solving* is pretty good, but not for beginners. It is intended for college students and advanced high school students. Its first few chapters give a good overview about how to investigate problems.

Two British authors, Judita Coffman and Tony Gardiner, have each written several interesting books about mathematical investigation, very suitable for teachers of all levels.

There are two great books that are quite elementary, more for middle-school/high school students (and their teachers), although I constantly use them for my college classes: *Mathematical Mosaic*, by Ravi Vakil, and *Solve This*, by Jim Tanton. Tanton also has many nice materials online. Visit his "St. Mark's Math Institute and poke around; make sure to look at the newsletters, which have the best stuff (http://www.stmarksschool.org/academics/mathematics/math-institute/index.aspx).

Math Circles There are math circles all over the country, but the oldest is (probably) in Boston, and the people who run it (Ellen and Robert Kaplan) are prolific writers. Visit

http://www.themathcircle.org/ to see some of their seminal work.

Lately, the West Coast has been very active, with the Mathematical Sciences Research Institute in Berkeley taking a leading role. Visit MSRI's web site, especially for students (http://www.msri.org/web/msri/areas-of-interest/students) and teachers

(http://www.msri.org/web/msri/areas-of-interest/educators) to get connected with math circles for students and teachers and various teacher-training programs. I am personally involved in many of these activities and would be happy to tell you more about them.

Also, MSRI has begun publishing books that are relevant to math circles in collaboration with the American Mathematical Society. The MSRI/AMS *Mathematical Circles Library* (http://library.msri.org/msri-mcl/index.html) has only published a few books so far, but one is very useful: *Circle in a Box*, by Sam Vandervelde, which is a very practical guide to starting a math circle from scratch.

The Eastern European Connection There is an enormous literature out there in Russian, Bulgarian, and Hungarian. Luckily, some of it has been translated. The AMS has published a number of books in its *Mathematical World* series. Especially notable are *Mathematical Circles (Russian Experience)* and the several *Kvant Selecta...* books, which are translated from the Russian journal *Kvant* (Quantum). I have been working on some translation projects, and recently helped to produce an American edition of *Children and Mathematics*, by Alexander Zvonkin, which is an account of the author's experiments working with very young kids in Moscow in the 1980's. This should be coming out soon, as part of the MSRI/AML Mathematical Circles Library.